Valuation of Inflation-Adjusted Floating Range Notes

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A B S T R A C T

Jarrow and Yildirim's (2003) model is adopted to derive pricing formulas for inflation-adjusted floating range notes. An inflation-adjusted floating range note has a similar structure of periodic interest payments as a regular floating range note, except that the principal would be adjusted with the change of an inflation index. This financial instrument provides investors with interest rate payments without inflation risks, maintaining the purchasing power of investment values when inflation is on the rise in the foreseeable future.

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1. Introduction

The Consumer Price Index (CPI) is an index to measure change in prices of a market basket of goods from one period to another. Its major function is to allow consumers to determine the degree to which their purchasing power is being eroded by price increase. During the current period of rising inflation due to rising prices of energies (oil and natural gas), minerals and farm produce, nominal investment returns have been eroded by inflation. Moreover, rising inflation seems to stay for the foreseeable future and raises the investors' concern for maintaining the purchasing power of their investment values. This leads to an increasing demand for inflation- linked derivatives.

For example, pension funds and insurance companies have been looking for inflation derivatives for long-term inflation protection. In addition, EUR inflation swaps have been experienced exponential growth in volume over the recent years, mainly driven by a variety of retail structures with short maturities. Many fixed income securities are now involved in inflation-adjusted features in some form.

Though conservative investors usually prefer investing in principal-protected financial products, nominally protected, the real purchasing power is in fact eroded during a long period of rising inflation. To avoid eroding the real purchasing power, the demand for inflation-adjusted financial instruments is on the rise. To follow the growth of the inflation derivatives market, we attempt to introduce a well-known derivative that incorporates an inflation-adjusted provision, an inflation-adjusted floating range note (hereafter, IFRN).[1]

An IFRN pays a floating coupon rate based on the value of some reference interest rate, observed at the beginning of each compounding period, with principal adjusted periodically for change in the CPI. With the provision on the inflation adjustment of principal, investors can earn real coupon payments which provide protection against rising inflation and maintain the real purchasing power.

In order to price IFRNs, we must suitably specify the dynamics of the CPI in a risk-neutral world. The various marco econometrics models are helpless when it comes to pricing inflation derivatives. Based on the Amin and Jarrow model (1991, AJ), Jarrow and Yildirim (2003, JY) have introduced a model that specifies the behaviors of the nominal and real instantaneous forward rates and the CPI (corresponding to domestic and foreign instantaneous forward rates and the exchange rate) in a risk-neutral world and used it to price treasury inflation-protected securities (TIPS) and inflation derivatives.

Based on the cross-currency LIBOR market model in Schlögl (2002), inflation has also been modelled by using market models. Mercurio (2005) presented two alternative market models to price zero coupon and year-on-year inflation-indexed swaps and inflation-indexed caps and floors.

The advantage of JY model is that the foreign-currency analogy gives an intuitive approach to viewing inflation. The rates specified in JY (2003) model are Gaussian processes which are flexible enough to price almost all inflation-linked derivatives in closed form. With this feature, hedging can be consistently conducted. In addition, the calibration procedure for parameters is well-specified in JY (2003), which makes this model more feasible and tractable. Therefore, we adopt the JY (2003) model to analytically price IFRNs.

The remaining of this paper is organized as follows. Section 2 reviews the JY model. The IFRNs are priced in Section 3. The conclusion is made in the last section.

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[1] The literatures on the valuation of regular floating range notes can be referred to Turnbull (1995), Navatte and Quittard-Pinon (1999), Nunes (2004), Eberlein and Kluge (2006) and Wu and Chen (2007).

2. Review of the Jarrow and Yildirim Model

The JY (2003) model is briefly reviewed in this section. We assume that trading takes place continuously in time over an interval $[0, \tau], 0 < \tau < \infty$. The uncertainty is described by a filtered spot martingale probability space $\Omega, \mathcal{F}, Q, \{\mathcal{F}_t\}_{t \in [0, \tau]}$ on which a d-dimensional standard Brownian motion $W(t) = (W_1(t), W_2(t), \ldots, W_d(t))$ is defined. The flow of information accruing to all the agents in the economy is represented by the filtration $\{\mathcal{F}_t\}_{t \in [0, \tau]}$ which satisfies the usual hypotheses.[2] Note that Q denotes the spot martingale probability measure. We list the notations with 'n' for nominal and 'r' for real as follows.

$X(t) =$ the time t inflation index (CPI).

$B_n(t, T) =$ the time t price in dollars of a nominal zero-coupon bond (ZCB) paying one dollar at time T.

$B_r(t, T) =$ the time t price in CPI units of a real ZCB paying one-unit of CPI at time T.

$B_{rx}(t, T) =$ $X(t)B_r(t, T)$, the time t nominal price in dollars of a real ZCB paying one-unit of CPI at time T.

$R(t, T; \Gamma_T) =$ the nominal forward LIBOR rate contracted at time t and applied to the period $[T, T + \delta_T]$ with $0 \leq t \leq T \leq T + \delta_T \leq T$.

$f_k(t, T) =$ the k forward interest rate contracted at time t for instantaneous borrowing and lending at time T $with$ $0 \leq t \leq T \leq \tau$, where $k \in \{n, r\}$.

$r_k(t) =$ $f_k(t, t)$, the k risk-free spot rate at time t.

$Q^T =$ the forward martingale measure with respective to the numéraire $B_n(\cdot, T)$.

The relationships between $R(t, T; \Gamma_T)$ and $B_n(t, T)$ and between $f_k(t, T)$ and $B_k(t, T)$ can be expressed as follows:

$$R(t, T; \Gamma_T) = \frac{1}{\delta_T}\left(B_n(t, T) - B_n(t, T + \Gamma_T)\right)/B_n(t, T + \Gamma_T) \tag{1}$$

$$B_K(t, T) = \exp\left(-\int_t^T f_k(t, u)du\right). \tag{2}$$

Based on the modeling mechanism used by AJ (1991), JY (2003) have derived the arbitrage-free evolution of the real and nominal forward rates, the prices of real and nominal ZCBs and the inflation index. We specify their results briefly as follows.

Proposition 2.1. The Jarrow and Yildirim Model under Q

The arbitrage-free processes of $f_k(t, u)$, $B_k(t, T)$, $B_{rx}(t, T)$ and $X(t)$ under the spot martingale measure Q are, respectively, given as follows:[3]

$$df_n(t, T) = \sigma_n(t, T) \cdot \sigma_{Bn}(t, T)dt + \sigma_n(t, T) \cdot dW(t), \tag{3}$$

$$df_r(t, T) = \sigma_r(t, T) \cdot \left(\sigma_{Br}(t, T) - \sigma_x(t)\right)dt + \sigma_r(t, T) \cdot dW(t), \tag{4}$$

$$\frac{dX(t)}{X(t)} = \left(r_n(t) - r_r(t)\right)dt + \sigma_x(t) \cdot dW(t), \tag{5}$$

$$\frac{dB_n(t, T)}{B_n(t, T)} = r_n(t)dt - \sigma_{Bn}(t, T) \cdot dW(t), \tag{6}$$

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[2] The filtration $\{F_t\}_{t \in [0, T]}$ is right continuous and F_0 contains all the Q-null sets of $\mathcal{F}$.

[3] For the detailed regular conditions, please refer to JY (2003).

$$\frac{dB_r(t,T)}{B_r(t,T)} = (r_r(t) + \sigma_x(t) \cdot \sigma_{Br}(t,T))dt - \sigma_{Br}(t,T) \cdot dW(t), \tag{7}$$

$$\frac{dB_{rx}(t,T)}{B_{rx}(t,T)} = r_n(t)dt + (\sigma_x(t) - \sigma_{Br}(t,T)) \cdot dW(t), \tag{8}$$

Where $\sigma_k(t,T)$, $\sigma_{Bk}(t,T)$ and $\sigma_x(t)$ are d-dimensional volatility vectors and the relationship between $\sigma_k(t,T)$ and $\sigma_{Bk}(t,T)$ is given as follows:

$$\sigma_{Bk}(t,T) = \int_t^T \sigma_k(t,s)ds.$$

By observing (3), *the drift restriction of the nominal forward rate for no-arbitrage* is still the same as the one given in the Heath, Jarrow and Morton (1992) model. For the real forward rate, the drift has one additional term, $(\sigma_r(t,T) \cdot \sigma_x(t))$, which specifies the instantaneous correlation between the inflation rate and the real forward rate. It is also observed that the drift terms of the real ZCB are augmented by the instantaneous correlations between the inflation rate and the real ZCB. For greater flexibility, the number of the random shocks, *d*, are not designated exactly, leaving the user to choose for simplicity and accuracy. The calibration procedure for parameters in Proposition 2.1 has been detailed in JY (2003).

Notice that dynamics in Proposition 2.1 are the stochastic processes under the spot martingale measure Q. It is sometimes convenient to know the dynamics under other forward martingale measures. The following proposition specifies a useful rule under which the stochastic processes are altered following the change of the underlying measure. This rule is convenient for deriving the pricing formulas of inflation derivatives.

Proposition 2.2. The Jarrow and Yildirim Model under Q^S

The arbitrage-free processes of $f_k(t,T), B_k(t,T), B_{rx}(t,T)$ and $X(t)$ under the forward martingale measure Q^S are, respectively, given as follows:

$$df_n(t,T) = \sigma_n(t,T) \cdot (\sigma_{Bn}(t,T) - \sigma_{Bn}(t,S))dt + \sigma_n(t,T) \cdot dW(t), \tag{9}$$

$$df_r(t,T) = \sigma_r(t,T) \cdot (\sigma_{Br}(t,T) - \sigma_x(t) - \sigma_{Bn}(t,S))dt + \sigma_r(t,T) \cdot dW(t), \tag{10}$$

$$\frac{dX(t)}{X(t)} = (r_n(t) - r_r(t) - \sigma_x(t) \cdot \sigma_{Bn}(t,S))dt + \sigma_{Bn}(t,T) \cdot dW(t), \tag{11}$$

$$\frac{dB_n(t,T)}{B_n(t,T)} = (r_n(t) + \sigma_{Bn}(t,T) \cdot \sigma_{Bn}(t,S))dt - \sigma_{Bn}(t,T) \cdot dW(t), \tag{12}$$

$$\frac{dB_r(t,T)}{B_r(t,T)} = (r_r(t) + (\sigma_x(t) + \sigma_{Bn}(t,S)) \cdot \sigma_{Br}(t,T))dt - \sigma_{Br}(t,T) \cdot dW(t), \tag{13}$$

$$\frac{dB_{rx}(t,T)}{B_{rx}(t,T)} = (r_n(t) - (\sigma_x(t) - \sigma_{Br}(t,T)) \cdot \sigma_{Bn}(t,S))dt + (\sigma_x(t) - \sigma_{Br}(t,T)) \cdot dW(t), \tag{14}$$

where $0 \le t \le min(S, T)$.[4]

Having briefly introduced the JY (2003) model, we next derive the pricing formulas of IFRNs in the following section.

3. Valuation of IFRNs

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[4] We employ *W (t)* to denote an independent *d*-dimensional standard Brownian motion under an arbitrary measure without causing any confusion.

The main purpose of this section is to derive the closed-form pricing formulas for IFRNs in the framework of the JY model. Before introducing a general IFRN contract, we define some notations as follows:

$P_0=$ the initial principal amount of an IFRN.

$\tau=$ the current time and the considered time flow is $0 \leq T_0 < \tau < T_1 < T_2 < \ldots < T_m$.

$D_i=$ the number of days during the year that contains the period $(T_i, T_{i+1}]$,where $(T_i, T_{i+1}]$ denotes the period from date T_i (excluding this date) up to and including date T_{i+1}.

$q =$ the number of days for the period $(\tau, T_1]$.

$N_i=$ the number of days for the period $(T_i, T_{i+1}]$.

$T_{0j}=$ the date $\tau + j$ for $j = 1, 2, \ldots, q$ $(T_{0_q} = T_1)$.

$T_{ij} =$ the date $T_i + j$ for $j = 1, 2, \ldots, N_i$ and $i = 1, 2, \ldots, m - 1(T_{i,N_i} = T_{i+1})$.

$C(\tau, T_i)=$ the time τ value of the i-th coupon paid at time T_i by an IFRN.

$K_{ij}^U (K_{ij}^L)=$ the upper (lower) bound of the range employed at time T_{ij}.

$\Delta_{ij} =$ the spread applied for the date T_{ij}.

$\Gamma_{ij}=$ the length of the compounding period (in years) of the reference interest rate observed at time T_{ij}.

$T_{ij}^* =$ the date $T_{ij} + \Gamma_{ij}$.

To lighten the notation, we disregard the third argument Γ_{ij} in the $R(t, T_{ij}; \Gamma_{ij})$, hoping without causing any confusion. To obtain a more general pricing formula, each reference LIBOR rate has its own compounding period.

Consider an IFRN, issued at time 0, with reset dates $\{T_0, T_1, \ldots, T_{m-1}\}$ and payment dates $\{T_1, \ldots, T_m\}$ where T_0 is the latest reset date, T_1 is the next reset date and T_n is the expiry date. The time flow can be explicitly presented as: $0 \leq T_0 < \tau < T_1 < T_2 < \ldots < T_m$. We define the coupon payments of this IFRN as follows.

Definition 3.1. *For an IFRN, the coupon at date T_1 is defined to be*

$$C(T_1, T_1) = M + P_0 \left[\sum_{j=1}^q (R(T_0, T_0) + \Delta_{0j}) \times J(T_{0j}) \right] / D_0 \qquad (15)$$

where M denotes the realized payoff during the period $(T_0, \tau]$ and

$$J(T_{0j}) = \begin{cases} 1 & ; K0_j^L \leq R(T_{0j}, T_{0j}) \leq K_{0j}^U \\ 0 & ; otherwise, \end{cases} \qquad (16)$$

for $j = 1, 2, \ldots, q$. The coupon at date T_{i+1}, for i=1,...,m-1 is given as follows:

$$C(T_{i+1}, T_{i+1}) = P_i \left[\sum_{j=1}^{N_i} (R(T_i, T_i) + \Delta_{ij}) \times J(T_{0j}) \right] / D_i \qquad (17)$$

Where

$$J(T_{ij}) = \begin{cases} 1; K_{ij}^L \leq R(T_{ij}, T_{ij}) \leq K_{ij}^U \\ 0; otherwise, \end{cases} \qquad (18)$$

for $j = 1, 2, \ldots, N_i$. The principal applied to each coupon is defined by

$$P_i = P_0 \frac{X(T_i)}{X(0)}, i = 0, 1, 2, \ldots, m \tag{19}$$

The final principal, denoted by Pm, is paid back at date Tm.

It is worth emphasizing that the pre-specified range $\left([K_{ij}^L, K_{ij}^U]\right)$ for each observed date is allowed to vary daily. In addition, to reflect different compensations arising from selling digital options with different ranges, the pre-specified spread (Δ_{ij}) is also permitted to change daily.

A specific feature of the IFRN is that its principal applied to each period is adjusted for change in the CPI index. This inflation-adjusted feature enables investors to avoid inflation risk and maintain the real purchasing powers of investment values. Therefore, IFRNs would be popularly accepted by investors in the present period of high inflation, which is expected to persist for years. If the principal remains constant, namely *P*, without being adjusted according to CPI index, IFRNs degenerate to a ordinary floating range notes (FRNs). In Section 4, we will present some numerical examples to show the effect of incorporating the adjustment of inflation.

Observe that an IFRN is a linear combination of four component financial products. It can be priced by first pricing these preliminary components, which are all inflation-adjusted: delayed digital options, delayed range digital options, delayed interest-or-nothing digital options and delayed interest-or-nothing range digital options. Each one of the four inflation-adjusted components will be priced sequentially in the following sections.

3.1 Inflation-Adjusted Delayed Digital Options (IDO)

An inflation-adjusted delayed digital call (put) option (IDC (IDP)) pays one unit of inflation-adjusted currency at maturity T_{i+1} if the reference interest rate $R(T_{ij}, T_{ij})$ that matured previously at time T_{ij} lies above (below) the strike rate K_{ij}. The final payoff of this option at time T_{i+1} is precisely given as follows:[5]

$$IDO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = \frac{X(T_i)}{X(0)} I(\theta R(T_{ij}, T_{ij}) \geq \theta K_{ij}) \tag{20}$$

where $R(T_{ij}, T_{ij})$ is the LIBOR rate for the period $[T_{ij}, T_{ij}^*]$, and $\theta = 1$ stands for a call and -1 for a put.

Theorem 3.1. *The value of an IDO at time τ is given as follows.*

$$IDO(\tau, T_{i+1}; T_{ij}; K_{ij}) = \frac{X_F(\tau, T_i)}{X(0)} B_n(\tau, T_{i+1}) \emptyset(\tau, T_i) N(\theta d(T_{ij})) \tag{21}$$

with

$$X_F(\tau, T_i) = \frac{X(\tau) B_r(\tau, T_i)}{B_n(\tau, T_i)}, \tag{22}$$

$$d(T_{ij}) = \frac{In\left(\frac{1+\Gamma_{ij} R(\tau, T_{ij})}{1+\Gamma_{ij} K_{ij}}\right) - \xi(\tau, T_{ij}) + \frac{1}{2} V(\tau, T_{ij})}{\sqrt{V(\tau, T_i)}}, \tag{23}$$

where $\xi(\tau, T_{ij}), \varphi(\tau, T_i)$ and $V(\tau, T_{ij})$ are defined, respectively, in (A.15), (A.13) and (A.16).

Proof. The proof is given in Appendix A.

Since the JY (2003) model is derived based on the AJ (1991) model, the JY model has a cross-currency analogy that nominal prices correspond to the domestic prices, real prices to foreign prices, and the CPI to the spot exchange rate. Thus, similar to the *interest rate parity*, $X_F(\tau, T_i)$ defined in

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[5] *I(A)* is an indicator function, defined as follows: $I(A) = \begin{cases} 1 & \text{if } A \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$

(22) stands for the time-T_i forward CPI index observed at time τ. Therefore, at time τ, the expected realized principal for time T_i is defined as follows:

$$\frac{X_F(\tau, T_i)}{X(0)}.$$

Thus, if inflation rises higher during the duration of an IDO, the adjusted principal would compensate the purchasing powers eroded by increasing prices.

$B_n(\tau, T_{i+1})$ in (21) represents the discount factor and the cumulative density function $N(\cdot)$ stands for the probability that the IDO will be in the money at maturity. As a market variable is observed at a previous time and used to calculate a payoff that occurs at a later time, this results in a timing adjustment in the pricing formula.[6] Observing the pricing formula (21), $\varphi(\tau, T_i)$ is a timing adjustment for the CPI index, observed at T_i and used to calculate the payoff at T_{i+1}.

3.2 Inflation-Adjusted Delayed Range Digital Options

An inflation-adjusted delayed range digital option (IDRO) is similar to an IDO except that the payment occurs as the reference rate lies inside a pre-specified range $[K_{ij}^U, K_{ij}^U]$. The final payoff. of a general IDRO at time T_{i+1} is defined as follows:

$$IDRO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = \frac{X(T_i)}{X(0)} I(K_{ij}^L \leq R(T_{ij}, T_{ij}) \leq K_{ij}^U). \tag{24}$$

Based on the property in probability measure theory, the IDRO payoff can be expressed in terms of two IDC payoffs. This means that (24) can be rewritten as follows:

$$IDRO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = IDC(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}^L) - IDC(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}^U).$$

This fact leads to the following theorem.

Theorem 3.2. *The time τ value of an IDRO is equal to*

$$IDRO(\tau, T_{i+1}; T_{ij}; K_{ij}) = IDC(\tau, T_{i+1}; T_{ij}; K_{ij}^L) - IDC(\tau, T_{i+1}; T_{ij}; K_{ij}^U). \tag{25}$$

3.3 Inflation-Adjusted Delayed Interest-or-Nothing Digital Options (IDIO)

An inflation-adjusted delayed interest-or-nothing digital call (put) option (IDIC (IDIP)) pays a floating interest payment $R(T_i, T_i)$ on the inflation-adjusted principal at maturity date T_{i+1} if the reference interest rate $R(T_{ij}, T_{ij})$ is above (below) a pre-specified strike rate K_{ij}. We state the contract formally by specifying its final payoff at time T_{i+1} as follows:

$$IDIRO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = \frac{X(T_i)}{X(0)} R(T_i, T_i) I(\theta R(T_{ij}, T_{ij}) \geq \theta K_{ij}),$$

where $R(T_i, T_i)$ is the LIBOR rate for the period $[T_i, T_i^*]$, and $\theta = 1$ stands for a call and -1 for a put. The pricing formula is derived in the following theorem.

Theorem 3.3. *The time τ value of an IDIO is given as follows:*

$$IDRO(\tau, T_{i+1}; T_{ij}; K_{ij}) = \frac{X_F(\tau, T_i)}{\Gamma_i X(0)} B_n(\tau, T_{i+1}) \left((1 + \Gamma_i R(\tau, T_i)) \emptyset^*(\tau, T_i) N\left(\theta d^*(T_{ij})\right) - \emptyset(\tau, T_i) N(\theta d(T_{ij})) \right). \tag{26}$$

with

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[6] Please refer to Hull (2006) for more detail about timing adjustment.

$$d^*(T_{ij}) = \frac{In\left(\frac{1 + \Gamma_{ij}R(\tau, T_{ij})}{1 + \Gamma_{ij}K_{ij}}\right) - \xi^*(\tau, T_{ij}) + \frac{1}{2}V(\tau, T_{ij})}{\sqrt{V(\tau, T_i)}}$$

where $\xi^*(\cdot, T_{ij})$, $\emptyset^*(\tau, T_i)$, $\emptyset(\tau, T_i)$, $V(\tau, T_{ij})$, $X_F(\tau, T_i)$ and $d(T_{ij})$ are defined, respectively, in (B.7), (B.10), (A.17), (A.20), (22) and (23).

Proof. The proof is given in Appendix B.

$X_F(\tau, T_i)/X(0)$, $\emptyset(\tau, T_i)$ and $B_n(\tau, T_{i+1})$ in (26) represent, respectively, the expected realized principal, a timing adjustment for the CPI index, and the discount factor. $\emptyset^*(\tau, T_i)$ is also a time adjustment for both of the LIBOR rate and CPI index, observed at time T_i and paid at time T_{i+1}. The cumulative density function $N(\cdot)$ stands for the probability that the IDIO will be in the money at maturity. If $\emptyset(\tau, T_i)$ and $\emptyset^*(\tau, T_i)$ in (26) are ignored, and $d(T_{ij})$ and $d^*(T_{ij})$ are seen to be the same, (26) can be rewritten as

$$\frac{X_F(\tau, T_i)}{X(0)}B_n(\tau, T_{i+1})R(\tau, T_i; \Gamma_i)N\left(d(T_{ij})\right),$$

which means that the IDIO pays the discounted LIBOR interset with inflation adjustment if the IDIO is in the money.

3.4 Inflation-Adjusted Delayed Interest-or-Nothing Range Digital Options

An inflation-adjusted delayed interest-or-nothing range digital option (IDIRO) is similar to an IDIO except that the payment occurs as the reference rate lies inside a pre-specified range $[K_{ij}^L, K_{ij}^U]$. The final payoff of an IDIRO is defined as follows:

$$IDIRO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = \frac{X(T_i)}{X(0)}R(T_i, T_i)I(K_{ij}^L \leq R(T_{ij}, T_{ij}) \leq K_{ij}^U).$$

Similar to IDROs, IDIROs can also be expressed in terms of two IDICs, i.e.

$$IDIRO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = IDIC(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}^L) - IDIC(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}^U).$$

Thus, the pricing formula of an IDIRO can be expressed in terms of the pricing formulas of IDICs, and the result is presented in the following theorem.

Theorem 3.4. *The time τ value of an IDIRO is equal to*

$$IDIRO(\tau, T_{i+1}; T_{ij}; K_{ij}) = IDIC(\tau, T_{i+1}; T_{ij}; K_{ij}^L) - IDIC(\tau, T_{i+1}; T_{ij}; K_{ij}^U). \quad (27)$$

3.5 Delayed Digital Options (DO)

An interest rate delayed digital call (put) option (DC (DP)) pays one currency unit at maturity T_{i+1} if the reference interest rate $R(T_{ij}, T_{ij})$ that matured previously at time T_{ij} lies above (below) the strike rate K_{ij}. The final payoff of this option at time T_{i+1} is precisely given as follows:

$$DO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = I(\theta R(T_{ij}, T_{ij}) \geq \theta K_{ij}).$$

Theorem 3.5. *The value of a DO at time τ is given as follows.*

$$DO(\tau, T_{i+1}; T_{ij}; K_{ij}) = B_n(\tau, T_{i+1})N\left(\theta \tilde{d}(T_{ij})\right) \quad (28)$$

with

$$\widetilde{d}(T_{ij}) = \frac{In\left(\frac{1+\Gamma_{ij}R(\tau,T_{ij})}{1+\Gamma_{ij}K_{ij}}\right)-\widetilde{\xi}(\tau,T_{ij})+\frac{1}{2}V(\tau,T_{ij})}{\sqrt{V(\tau,T_i)}}, \tag{29}$$

where

$$\widetilde{\xi}(\tau,T_{ij}) = \int_{\tau}^{T_{ij}} \mu_h(u,T_{ij})du$$

and $\mu_h(u,T_{ij})$ is defined in (A.12).

Proof. The proof of Theorem 3.5 is available upon request from the authors.

3.6 Delayed Range Digital Options

A delayed range digital option (DRO) is similar to an DO except that the payment occurs as the reference rate lies inside a pre-specified range $[K_{ij}^L, K_{ij}^U]$. The final payoff of a general DRO at time T_{i+1} is defined as follows:

$$DRO(T_{i+1},T_{i+1};T_{ij};K_{ij}) = I(K_{ij}^L \leq R(T_{ij},T_{ij}) \geq K_{ij}^U). \tag{30}$$

Based on the property in probability measure theory, an DRO payoff can be expressed in terms of two DC payoffs. Equation (30) can then be rewritten as follows:

$$DRO(T_{i+1},T_{i+1};T_{ij};K_{ij}) = DC(T_{i+1},T_{i+1};T_{ij};K_{ij}^L) - DC(T_{i+1},T_{i+1};T_{ij};K_{ij}^U).$$

This result leads to the following theorem.

Theorem 3.6. *The time τ value of a DRO is equal to*

$$DRO(\tau,T_{i+1};T_{ij};K_{ij}) = DC(\tau,T_{i+1};T_{ij};K_{ij}^L) - DC(\tau,T_{i+1};T_{ij};K_{ij}^U). \tag{31}$$

Remark 3.1. *For an DO, if the maturity date T_{ij} of its reference rate equals T_{i+1} which is also the maturity date of the DO, then an DO becomes an ordinary digital option without delaying its payoff. Similarly, as $T_{ij} = T_{i+1}$, an DRO also becomes an ordinary digital range option.*

3.7 Delayed Interest-or-Nothing Digital Options

A delayed interrest-or-nothing digital call (put) option (DIC (DIP)) pays a floating interest payment $R(T_i,T_i)$ at maturity date T_{i+1} if the reference interest rate $R(t_{ij},T_{ij})$ is above (below) a pre-specified strike rate K_{ij}. The final payoff of a general DIO at time T_{i+1} (with $\theta = 1$ standing for DIC and $\theta = -1$ standing for DIP) is defined as follows:

$$DIO(T_{i+1},T_{i+1};T_{ij};K_{ij}) = R(T_i,T_i)I(\theta R(T_{ij},T_{ij}) \geq \theta K_{ij}). \tag{32}$$

Theorem 3.7. *The value of a DIO at time τ is given as follows:*

$$DIO(\tau,T_{i+1};T_{ij};K_{ij}) = \frac{B_n(\tau,T_{i+1})}{\Gamma_i}\left((1+\Gamma_i R(\tau,T_i))\widehat{\varphi}(\tau,T_i)N\left(\theta\widehat{d}(T_{ij})\right) - N\left(\theta\widehat{d}(T_{ij})\right)\right), \tag{33}$$

where

$$\widehat{\varphi}(\tau,T_i) = exp\left(\int_{\tau}^{T_i} \mu_{\widehat{g}}(u,T_i)du\right),$$

$$\widehat{d}(T_{ij}) = \frac{In\left(\frac{1+\Gamma_{ij}R(\tau,T_{ij})}{1+\Gamma_{ij}K_{ij}}\right) - \widetilde{\xi}(\tau,T_{ij}) + \frac{1}{2}V\left(\tau,T_{v(\tau,T_{ij})ij}\right)}{\sqrt{V(\tau,T_i)}},$$

$$\widehat{\xi}(\tau,T_i) = \int_{\tau}^{T_{ij}}(\mu_h(u,T_{ij}) + \widehat{\eta}(u,T_{ij})\cdot v_h(u,T_{ij}))du,$$

$$\hat{\eta}(u, T_{ij}) = \begin{cases} v_{\hat{g}}(t, T_i) & 0 \leq t \leq T_i, \\ 0 & T_i \leq t \leq T_{ij}, \end{cases}$$

$$\mu_{\hat{g}}(t, T_i) = (\sigma_{Bn}(t, T_{i+1}) - \sigma_{Bn}(t, T_I^*)) \cdot (\sigma_{Bn}(t, T_i) - \sigma_{Bn}(t, T_I^*))$$

$$v_{\hat{g}}(t, T_i) = (\sigma_{Bn}(t, T_I^*) - \sigma_{Bn}(t, T_i)),$$

and $V(\tau, T_{ij})$, $\mu_g(t, T_i)$, $v_g(t, T_i)$ are defined, respectively, in (A.20), (A.11) and (A.14).

Proof. The proof of Theorem 3.7 is available upon request from the authors.

3.8 Delayed Interest-or-Nothing Range Digital Options

A delayed interrest-or-nothing range digital call (put) option (DIRO) pays a floating interest payment $R(T_i, T_i)$ at maturity date T_{i+1} if the reference interest rate $R(T_{ij}, T_{ij})$ lies within a pre-specified range $[K_{ij}^L, K_{ij}^U]$. The final payoff of the DIRO is defined as follows:

$$DIRO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = R(T_i, T_i)I(K_{ij}^L \leq R(T_{ij}, T_{ij}) \leq K_{ij}^U). \tag{34}$$

Theorem 3.8. *The value of a DIRO at time τ is given as follows:*

$$DIRO(\tau, T_{i+1}; T_{ij}; K_{ij}) = DIC(\tau, T_{i+1}; T_{ij}; K_{ij}^L) - DIC(\tau, T_{i+1}; T_{ij}; K_{ij}^U) \text{ where } DIC(\tau, T_{i+1}; T_{ij}; K_{ij}^L)$$
and $DIC(\tau, T_{i+1}; T_{ij}; K_{ij}^U)$ are defined in (33).

3.9 Valuation of IFRNs and FRNs

As indicated in Definition 3.1 and 3.2, IFRNs and FRNs are composed of various digital options presented in the above subsections. With the above preliminary theorems, the pricing formulas of IFRNs and FRNs can be derived and given in the following theorem.

Theorem 3.9. *For an IFRN as defined in Definition 3.1., the time τ value of its first coupon is equal to:*

$$C(\tau, T_1) = MB_n(\tau, T_1) + P_0 \sum_{j=1}^{q} [(R(T_0, T_0) + \Delta_{0j}) \times DRO(\tau, T_1; T_{0j}; K_{0j})] / D_0.$$

The time τ value of other coupons is given as follows:

$$C(\tau, T_{i+1}) = P_0 \sum_{j=1}^{N_i} [IDIRO(\tau, T_{i+1}; T_{ij}; K_{ij}) + \Delta_{0j} \times IDRO(\tau, T_{i+1}; T_{ij}; K_{ij})] / D_i,$$

for $i = 1, 2, \dots, m-1$. The present value of the final principal at time T_n is equal to:

$$P_0 \frac{X_F(\tau, T_n)}{X(0)} B_n(\tau, T_n).$$

Thus, the time τ value of an IFRN is equal to:

$$IFRN = \sum_{i=1}^{m} C(\tau, T_i) + P_0 \frac{X_F(\tau, T_n)}{X(0)} B_n(\tau, T_n). \tag{35}$$

Theorem 3.10. *For an FRN as defined in Definition 3.2, the time τ value of its first coupon is equal to:*

$$C(\tau, T_1) = MB_n(\tau, T_1) + P_0 \sum_{j=1}^{q} \left[(R(T_0, T_0) + \Delta_{0j}) \times DRO(\tau, T_1; T_{0j}; K_{0j}) \right] / D_0.$$

The time τ value of other coupons is given as follows:

$$C(\tau, T_{i+1}) = P_0 \sum_{j=1}^{N_i} \left[IDIRO(\tau, T_{i+1}; T_{ij}; K_{ij}) + \Delta_{ij} \times IDRO(\tau, T_{i+1}; T_{ij}; K_{ij}) \right] / D_i,$$

for $i = 1, 2, \ldots, m - 1$. The present value of the final principal at time T_n is equal to:

$$P_0 \frac{X_F(\tau, T_n)}{X(0)} B_n(\tau, T_n).$$

Thus, the time τ value of an IFRN is equal to:

$$IFRN = \sum_{i=1}^{m} C(\tau, T_i) + P_0 \frac{X_F(\tau, T_n)}{X(0)} B_n(\tau, T_n). \tag{36}$$

IFRNs can be priced analytically with Theorem 3.7. The parameters in IFRN pricing formula (37) can also be calibrated according to the calibration mechanism provided by JY (2003). Our model is thus tractable and feasible for market participants in inflation derivatives markets.

4. Conclusions

The pricing formula of an IFRN has been analytically derived. With functioning as a floating range note, an IFRN provides investors interest payments periodically without inflation risk. Via inflation-adjusted provision, investment values would not be eroded by rising prices of goods and services, and thereby maintaining the real purchasing power.

The inflation derivatives market has been developing at a rapid pace and the market demand for structured inflation products is also increasing. This growing trend will definitely draw more attention to the pricing of various inflation-linked derivatives. Thus, an IFRN should be a welcome inflation derivative which most likely becomes a popular financial instrument, especially in the presence of persistent high inflation for years to come.

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Appendix A: Proof of Theorem 3.1

This appendix is to derive the pricing formula for the IDO specified in Theorem 3.1. We first price the IDC under the forward measure Q^{Ti+1} as follows:

$$IDC(\tau, T_{i+1}; T_{ij}; K_{ij}) = B_n(\tau, T_{i+1})E^{Q^{Ti+1}}\left(\frac{X(T_i)}{X(0)}I(R(T_{ij}; T_{ij}) > K_{ij})|F_\tau\right). \tag{A.1}$$

According to (1), we can rewrite (A.1) as follows:

$$(A.1) = \frac{B_n(\tau, T_{i+1})}{X(0)}E^{Q^{Ti+1}}(X(T_i)I(B_n(T_{ij}, T_{ij}^*) < \bar{K}_{ij})|F_\tau), \tag{A.2}$$

where

$$\bar{K}_{ij} = \frac{1}{1+\Gamma_{ij}K_{ij}}. \tag{A.3}$$

For computing (A.2), we further rewrite the expectation part in (A.2) as follows:

$$E^{Q^{Ti+1}}\left(\frac{X(T_i)B_r(T_i, T_i)}{B_n(T_i, T_{i+1})}\frac{B_n(\tau, T_{i+1})}{B_n(T_i, T_i)}I\left(\frac{B_n(T_{ij}, T_{ij}^*)}{B_n(T_i, T_{i+1})} < \bar{K}_{ij}\frac{B_n(T_{ij}, T_{ij})}{B_n(T_i, T_{i+1})}\right)|F_\tau\right), \tag{A.4}$$

Define the following notations:

$$\alpha(t, T_i) = \frac{X(t)B_r(t, T_i)}{B_n(t, T_{i+1})},$$

$$b(t, T_i) = \frac{B_n(t, T_{i+1})}{B_n(t, T_i)},$$

$$c(t, T_{ij}) = \frac{B_n(t, T_{ij}^*)}{B_n(t, T_{i+1})},$$

$$e(t, T_{ij}) = \frac{B_n(t, T_{ij})}{B_n(t, T_{i+1})}.$$

Based on the *Itô* Lemma and the changing-measure technique, the dynamics of $\alpha(t, T_i)$, $b(t, T_i)$, $c(t, T_{ij})$ and $e(t, T_{ij})$ are given as follows:

$$\frac{d\alpha(t,T_i)}{\alpha(t,T_i)} = \left(\sigma_x(t) + \sigma_{B_n}(t, T_{i+1}) - \sigma_{B_r}(t, T_i) \right) \cdot dW(t), \tag{A.5}$$

$$\frac{db(t,T_i)}{b(t,T_i)} = \left\| \sigma_{B_n}(t, T_{i+1}) - \sigma_{B_n}(t, T_i) \right\|^2 dt - (\sigma_{B_n}(t, T_{i+1}) - \sigma_{B_n}(t, T_i)) \cdot dW(t), \tag{A.6}$$

$$\frac{dc(t,T_{ij})}{c(t,T_{ij})} = \left(\sigma_{B_n}(t, T_{i+1}) - \sigma_{B_n}(t, T_{ij}^*) \right) \cdot dW(t), \tag{A.7}$$

$$\frac{de(t,T_{ij})}{e(t,T_{ij})} = \left(\sigma_{B_n}(t, T_{i+1}) - \sigma_{B_n}(t, T_{ij}) \right) \cdot dW(t). \tag{A.8}$$

We further define $g(t, T_i) = \alpha(t, T_i) b(t, T_i)$ and $h(t, T_{ij}) = c(t, T_{ij})/e(t, T_{ij})$ and their dynamics can be derived via *Itô* Lemma as follows:

$$\frac{dg(t,T_i)}{g(t,T_i)} = \mu_g(t, T_i) dt + v_g(t, T_i) \cdot dW(t), \tag{A.9}$$

$$\frac{dh(t,T_{ij})}{h(t,T_{ij})} = \mu_h(t, T_{ij}) dt + v_h(t, T_{ij}) \cdot dW(t) \tag{A.10}$$

where

$$\mu_g(t, T_i) = \left(\sigma_{B_n}(t, T_{i+1}) - \sigma_{B_n}(t, T_i) \right) \cdot (\sigma_{B_r}(t, T_i) - \sigma_{B_n}(t, T_i) - \sigma_x(t)), \tag{A.11}$$

$$\mu_h(t, T_{ij}) = \left(\sigma_{B_n}(t, T_{ij}^*) - \sigma_{B_n}(t, T_{ij}) \right) \cdot (\sigma_{B_n}(t, T_{i+1}) - \sigma_{B_n}(t, T_{ij})), \tag{A.12}$$

$$v_g(t, T_i) = \left(\sigma_x(t) + \sigma_{B_n}(t, T_i) - (\sigma_{B_r}(t, T_i) \right), \tag{A.13}$$

$$v_h(t, T_{ij}) = \left(\sigma_{B_n}(t, T_{ij}) - (\sigma_{B_n}(t, T_{ij}^*) \right). \tag{A.14}$$

Thus, (A.4) can be rewritten as follows:

$$(A.4) = E^{Q^{Ti+1}}\left(g(t, T_i)\, I(h(T_{ij}; T_{ij}) < \bar{K}_{ij}) | F_\tau \right)$$
$$= g(\tau, T_i) \exp\left(\int_T^{T_i} \mu_g(t, T_i)\, |du \right) E^{Q^{Ti+1}} \left(\frac{d\bar{Q}^{Ti+1}}{d\bar{Q}^{Ti+1}} I(h(T_{ij}; T_{ij}) < \bar{K}_{ij}) | F_\tau \right),$$
$$= g(\tau, T_i) \emptyset(\tau, T_i) P^{\bar{Q}^{Ti+1}}(h(T_{ij}; T_{ij}) < \bar{K}_{ij} | F_\tau), \tag{A.15}$$

where the Radon-Nikodý'm derivative is defined as follows:

$$\frac{d\bar{Q}^{Ti+1}}{d\bar{Q}^{Ti+1}} = exp(-\frac{1}{2}\int_T^{T_{ij}} \|\eta(u, T_{ij})\|^2 du + \int_T^{T_{ij}} \eta(u, T_{ij}) \cdot dW(t), \tag{A.16}$$

and

$$\emptyset(\tau, T_i) = exp \int_T^{T_i} \mu_g(t, T_i) du, \tag{A.17}$$

$$\eta(t, T_{ij}) = \begin{cases} v_g(t, T_i) & 0 \leq t \leq T_i, \\ 0 & T_i < t \leq T_{ij}. \end{cases}$$

The Radon-Nikody´m derivative, (A.12), can be used to define a new probability measure as follows:

$$\bar{Q}^{Ti+1}(A) = \int_A \frac{d\bar{Q}^{Ti+1}}{d\bar{Q}^{Ti+1}} dQ^{Ti+1}(\omega) \, for \, all \, A \epsilon \mathcal{F}.$$

According to the Girsanov's Theorem, a new d-dimensional Brownian, $W\bar{Q}^{Ti+1}(t)$, under the new probability measure $\bar{Q}^{Ti+1}$ can be defined as follows:

$$W^{\bar{Q}^{Ti+1}}(t) = W^{\bar{Q}^{Ti+1}}(t) - \int_\tau^t \eta(u, T_{ij}) \, du.$$

Thus, under $\bar{Q}^{Ti+1}$, the dynamics of $h(t, T_{ij})$ is given as follows:

$$\frac{dh(t, T_{ij})}{h(t, T_{ij})} = (\mu_h(t, T_{ij})dt + \eta(t, T_{ij}) \cdot v_h(t, T_{ij}))dt + v_h(t, T_{ij}) \cdot dW(t),$$

The solution of $h(T_{ij}, T_{ij})$ is given as follows:

$$h(T_{ij}, T_{ij}) = h(\tau, T_{ij})exp(\xi(\tau, T_{ij}) - \frac{1}{2}V(\tau, T_{ij}) + Z_{ij}) \tag{A.18}$$

where

$$Z_{ij} = \int_\tau^{T_{ij}} v_h(u, T_{ij}) \cdot dW(u) \sim N(0, V(\tau, T_{ij})$$

$$\xi(\tau, T_{ij}) = \int_\tau^{T_{ij}} (\mu_h(u, T_{ij}) + \eta(t, T_{ij}) \cdot v_h(u, T_{ij})) \, du \tag{A.19}$$

$$V(\tau, T_{ij}) = \int_\tau^{T_{ij}} \|v_h(u, T_{ij})\|^2 du \tag{A.20}$$

Thus, the probability in (A.11) can be computed as follows:

$$P^{\bar{Q}^{Ti+1}}\left(h(T_{ij}, T_{ij}) < \bar{K}_{ij}|F_\tau\right) = P^{\bar{Q}^{Ti+1}}\left(h(T_{ij}, T_{ij})exp(\xi(\tau, T_{ij}) - \frac{1}{2}V(\tau, T_{ij}) + Z_{ij}\right)$$
$$< \bar{K}_{ij}|F_\tau) = N(d(T_{ij}))$$

where

$$d(T_{ij}) = \frac{ln\left(\frac{\bar{K}_{ij}}{h(\tau, T_{ij})}\right) - \xi(\tau, T_{ij}) + \frac{1}{2}V(\tau, T_{ij})}{\sqrt{V(\tau, T_{ij})}}. \tag{A.21}$$

With the above computation, the pricing formula for the IDC can be derived as follows:

$$IDC(\tau, T_{i+1}; T_{ij}; K_{ij}) = \frac{X(\tau)B_r(t, T_{ij})}{X(0)B_n(t, T_{ij})} B_n(t, T_{i+1})\emptyset(\tau, T_i)N\left(d(T_{ij})\right). \tag{A.22}$$

As to the pricing formula for the IDP, it can be derived similarly to the IDC case.

Appendix B: Proof of Theorem 3.3

This appendix is to derive the pricing formula for IDIO specified in Theorem 3.3. We first price the IDIC

under the forward measure Q^{Ti+1} as follows:

$$IDIC(\tau, T_{i+1}; T_{ij}; K_{ij}) = B_n(\tau, T_{i+1})E^{Q^{Ti+1}}\left(\frac{X(T_i)}{X(0)}R(T_i; T_i)I(R(T_{ij}; T_{ij})) > K_{ij})|F_\tau\right).$$
(B.1)

According to (1), we can rewrite (B.1) as follows:

$$(B.1) = \frac{B_n(\tau, T_{i+1})}{X(0)}E^{Q^{Ti+1}}(X(T_i)(\frac{1-B_n(T_i, T_i^*)}{\Gamma_i B_n(T_i, T_i^*)})I(B_n(T_{ij}, T_{ij}^*) < \overline{K}_{ij})|F_\tau),$$

$$= \frac{B_n(\tau, T_{i+1})}{\Gamma_i X(0)}E^{Q^{Ti+1}}(\frac{X(T_i)B_r(T_i, T_i)}{B_n(T_i, T_i^*)})I(B_n(T_{ij}, T_{ij}^*) < \overline{K}_{ij})|F_\tau) - \frac{B_n(\tau, T_{i+1})}{\Gamma_i X(0)}E^{Q^{Ti+1}}(X(T_i) I(B_n(T_{ij}, T_{ij}^*) < \overline{K}_{ij})|F_\tau).$$

(B.2)

The second part in (B.2) can be obtained by Appendix A as follows:

$$\frac{B_n(\tau, T_{i+1})}{\Gamma_i X(0)}E^{Q^{Ti+1}}(X(T_i) I(B_n(T_{ij}, T_{ij}^*) < \overline{K}_{ij})|F_\tau) = \frac{IDC(\tau, T_{i+1}; T_{ij}; K_{ij})}{\Gamma_i}.$$
(B.3)

Next, we rewrite the first part in (B.2) as follows:

$$\frac{B_n(\tau, T_{i+1})}{\Gamma_i X(0)}E^{Q^{Ti+1}}(g^*(T_i, T_i) I(h(T_{ij}, T_{ij}) < \overline{K}_{ij})|F_\tau)$$
(B.4)

where $h(T_{ij}, T_{ij})$ is defined in (A.14) and $g^*(t, T_i)$ is defined as follows:

$$g^*(t, T_i) = \frac{X(t)B_r(t, T_i)}{B_n(t, T_{i+1})}\frac{B_n(t, T_{i+1})}{B_n(t, T_i^*)}.$$

Similar to the deriving process of IDC in Appendix A, (B.4) can be derived by replacing $g(t, T_i)$ with $g^*(t, T_i)$ and the result is given as follows:

$$(B.4) = \frac{X(\tau)B_r(\tau, T_i)}{X(0)B_n(t, T_i)}\frac{B_n(\tau, T_i)}{\Gamma_i B_n(\tau, T_i^*)}\emptyset(\tau, T_i)B_n(\tau, T_i)N\left(d^*(T_{ij})\right).$$
(B.5)

where

$$d^*(T_{ij}) = \frac{ln\left(\frac{1+\Gamma_{ij}R(\tau, T_{ij})}{1+\Gamma_{ij}K_{ij}}\right) - \xi^*(\tau, T_{ij}) + \frac{1}{2}V(\tau, T_{ij})}{\sqrt{V(\tau, T_{ij})}},$$
(B.6)

$$\xi^*(\tau, T_{ij}) = \int_\tau^{T_{ij}}(\mu_h(u, T_{ij}) + \eta^*(u, T_{ij}) \cdot v_h(u, T_{ij})) du,$$
(B.7)

$$\eta^*(u, T_{ij}) = \begin{cases} v_{g^*}(t, T_i) & 0 \le t \le T_i, \\ 0 & T_i < t \le T_{ij}, \end{cases}$$
(B.8)

$$v_{g^*}(t, T_i) = (\sigma_x(t) + \sigma_{B_n}(t, T_i^*) - (\sigma_{B_r}(t, T_i),$$
(B.9)

$$\emptyset^*(\tau, T_i) = \exp(\int_T^{T_i} \sigma_{B_n}(u, T_{i+1}) - \sigma_{B_n}(u, T_i^*)) \cdot (\sigma_{B_r}(u, T_i) - \sigma_{B_n}(u, T_i^*) - \sigma_x(u))).$$
(B.10)

With the above derivation, IDIC can be derived as follows:

$$IDIC(\tau, T_{i+1}; T_{ij}; K_{ij}) = (B.5) - (B.3).$$

As to the pricing formula for the IDIP, it can be derived similarly to the IDIC case.